

# CYCLICITY AND $R$ -MATRICES

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ABSTRACT. Let  $S_1, \dots, S_N$  simple finite-dimensional modules of a quantum affine algebra. We prove that if  $S_i \otimes S_j$  is cyclic for any  $i < j$  (i.e. generated by the tensor product of the highest weight vectors), then  $S_1 \otimes \dots \otimes S_N$  is cyclic. The proof is based on the study of  $R$ -matrices.

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## 1. INTRODUCTION

Let  $q \in \mathbb{C}^*$  which is not a root of unity and let  $\mathcal{U}_q(\mathfrak{g})$  be the quantum affine algebra corresponding to an affine Kac-Moody algebra  $\mathfrak{g}$ . Let  $\mathcal{F}$  be the category of finite-dimensional representations of the algebra  $\mathcal{U}_q(\mathfrak{g})$ . The algebra  $\mathcal{U}_q(\mathfrak{g})$  being a Hopf algebra, the category  $\mathcal{F}$  has a tensor structure. This tensor category has been studied from many points geometric, algebraic, combinatorial perspectives in connections to various fields, see [MO, KKKO] for recent important developments, [H4] for a recent review and [CWY] for a very recent point of view in the context of physics.

$\mathcal{F}$  has a very intricate structure. It is not known if the category  $\mathcal{F}$  is wild or tame. All simple objects in  $\mathcal{F}$  have been classified by Chari-Pressley (see [CP2]), although many basic questions are still open, such as the dimension of the simple objects of  $\mathcal{F}$  or the classification of its indecomposable objects. The aim of the present paper is to answer to one of these basic questions, namely the compatibility of tensor product with cyclicity.

A representation in  $\mathcal{F}$  has a weight space decomposition with respect to the underlying finite type quantum group  $\mathcal{U}_q(\bar{\mathfrak{g}}) \subset \mathcal{U}_q(\mathfrak{g})$ . A module in  $\mathcal{F}$  is said to be *cyclic* if it is generated by a highest weight vector with respect to  $\mathcal{U}_q(\bar{\mathfrak{g}})$ .

The main result of the paper is the following.

**Theorem 1.1.** *For  $S_1, \dots, S_N$  simple finite-dimensional modules of a quantum affine algebra such that  $S_i \otimes S_j$  is cyclic for any  $i < j$ , the module  $S_1 \otimes \dots \otimes S_N$  is cyclic.*

A proof is given in this paper and is based on a detailed study of certain intertwining operators in the category  $\mathcal{F}$  derived from  $R$ -matrices.

A reader less familiar with the representation theory of quantum affine algebras may wonder why such a result is non trivial. Indeed, in the category of finite-dimensional representations of  $\mathcal{U}_q(\bar{\mathfrak{g}})$  or of  $\bar{\mathfrak{g}}$ , a tensor product of simple representations is never cyclic, unless one of the modules in question is one-dimensional. But in the category  $\mathcal{F}$ , a tensor product of two simple modules is generically simple and so cyclic. Moreover the category  $\mathcal{F}$  is non semi-simple and there are "many" non simple cyclic tensor products.

For instance, an important result proved in [C, Kas, VV] (see Theorem 2.4 below) gives a sufficient condition for a tensor product of fundamental representations in  $\mathcal{F}$  to be cyclic. This implies a particular case of our Theorem 1.1 for such tensor products. This cyclicity result has been extended in [C] for Kirillov-Reshetikhin modules and in [JM] for some more general simple modules in the  $sl_2$ -case.

Our Theorem 1.1 may also be seen as a generalization of the main result of [H3] (if  $S_i \otimes S_j$  is simple for any  $i < j$ , then the module  $S_1 \otimes \dots \otimes S_N$  is simple). In fact Theorem 1.1 implies this result as a cyclic module whose dual is cyclic is simple (see [CP1, Section 4.10]).

The result in the present paper is much more general, even in the  $\hat{sl}_2$ -case. Our proof is valid for arbitrary simple objects of  $\mathcal{F}$  and for arbitrary  $\mathfrak{g}$ .

Our result is stated in terms of the tensor structure of  $\mathcal{F}$ . Thus, it is purely representation theoretical. But we have additional motivations.

Although the category  $\mathcal{F}$  is not braided,  $\mathcal{U}_q(\mathfrak{g})$  has a *universal  $R$ -matrix* in a completion of the tensor product  $\mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g})$ . In general the universal  $R$ -matrix can not be specialized to finite-dimensional representations, but when  $V \otimes V'$  is cyclic, we get a well-defined morphism

$$\mathcal{I}_{V,V'} : V \otimes V' \rightarrow V' \otimes V.$$

In fact, it is expected that the localization of the poles of  $R$ -matrices (more precisely of meromorphic deformations of  $R$ -matrices, see below) is related to the irreducibility and to the cyclicity of tensor product of simple representations. This is an important question from the point of view of mathematical physics. The interest for such  $R$ -matrices was revived recently with the fundamental work of Maulik-Okounkov on stable envelopes [MO].

The existence of a cyclic tensor product  $A \otimes B$  is related to the existence of certain non-split short exact sequences in  $\mathcal{F}$  of the form

$$0 \rightarrow V \rightarrow A \otimes B \rightarrow W \rightarrow 0.$$

These are crucial ingredients for categorification theory as they imply remarkable relations in the Grothendieck ring of the category  $\mathcal{F}$  of the form

$$[A] \times [B] = [V] + [W].$$

Examples include the  $T$ -systems [N, H1] or certain Fomin-Zelevinsky cluster mutations [HL1, KKKO, HL2].

The classification of indecomposable object in  $\mathcal{F}$  is not known, even in the  $\hat{sl}_2$ -case. Although by [BN] the maximal cyclic modules are known to be the Weyl modules (the cyclic tensor product of fundamental representations), the classification of cyclic modules is unknown. The main result of the present paper is a first step in this direction as it provides a factorization into a tensor product of simple modules of a large family of cyclic modules. Such problems of factorization into simple or prime simple modules (i.e. which can not be written as a tensor product of non trivial simple objects) is an open problem for  $\mathcal{F}$  and is one of the main motivation in [HL1]. This is also related to the problem of monoidal categorification of cluster algebras.

The results of the present paper are used by Gurevich [G] in his proof of one direction of the Gan-Gross-Prasad conjecture for  $p$ -adic groups  $GL_N$ .

The paper is organized as follows. In Section 2 we give reminders on quantum affine algebra, its category  $\mathcal{F}$  of finite-dimensional representations and the corresponding ( $q$ )-character theory. We recall a result on cyclicity of tensor products of fundamental representations (Theorem 2.4), a result on tensor product of ( $\ell$ )-weight vectors (Theorem 2.3) and the main properties of intertwiners and their deformations. In Section 3 we reduce the problem by considering certain remarkable subcategories which have a certain compatibility property with characters (Theorem 3.5). In Section 4 we end the proof of Theorem 1.1.

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## 2. FINITE-DIMENSIONAL REPRESENTATIONS OF QUANTUM AFFINE ALGEBRAS

Following [H3], we recall the main definitions and properties of finite-dimensional representations of quantum affine algebras. In particular we discuss the corresponding ( $q$ )-character theory, the intertwiners derived from  $R$ -matrices, their deformations and an important cyclicity result.

**2.1. Quantum affine algebras.** All vector spaces, algebras and tensor products are defined over  $\mathbb{C}$ .

Let  $C = (C_{i,j})_{0 \leq i,j \leq n}$  be a *generalized Cartan matrix* [Kac], i.e. for  $0 \leq i, j \leq n$  we have  $C_{i,j} \in \mathbb{Z}$ ,  $C_{i,i} = 2$ , and for  $0 \leq i \neq j \leq n$  we have  $C_{i,j} \leq 0$ , ( $C_{i,j} = 0 \Leftrightarrow C_{j,i} = 0$ ). We suppose that  $C$  is *indecomposable*, i.e. there is no proper  $J \subset \{0, \dots, n\}$  such that  $C_{i,j} = 0$  for any  $(i, j) \in J \times (\{0, \dots, n\} \setminus J)$ . Moreover we suppose that  $C$  is of *affine type*, i.e. all proper principal minors of  $C$  are strictly positive and  $\det(C) = 0$ . By the general theory in [Kac],  $C$  is *symmetrizable*, that is there is a diagonal matrix with rational coefficients  $D = \text{diag}(r_0, \dots, r_n)$  such that  $DC$  is symmetric. Fix  $h \in \mathbb{C}$  satisfying  $q = e^h$ . Then  $q^p = e^{hp}$  is well-defined for any  $p \in \mathbb{Q}$ .

The *quantum affine algebra*  $\mathcal{U}_q(\mathfrak{g})$  is defined by generators  $k_i^{\pm 1}$ ,  $x_i^{\pm}$  ( $0 \leq i \leq n$ ) and relations

$$k_i k_j = k_j k_i, \quad k_i x_j^{\pm} = q^{(\pm r_i C_{i,j})} x_j^{\pm} k_i, \quad [x_i^+, x_j^-] = \delta_{i,j} \frac{k_i - k_i^{-1}}{q^{r_i} - q^{-r_i}},$$

$$\sum_{p=0 \dots 1-C_{i,j}} (-1)^p (x_i^\pm)^{(1-C_{i,j}-p)} x_j^\pm (x_i^\pm)^{(p)} = 0 \text{ (for } i \neq j),$$

where we denote  $(x_i^\pm)^{(p)} = (x_i^\pm)^p / [p]_q r_i!$  for  $p \geq 0$ . We use the standard  $q$ -factorial notation  $[p]_q! = [p]_q [p-1]_q \cdots [1]_q = (q^p - q^{-p})(q^{p-1} - q^{1-p}) \cdots (q - q^{-1})(q - q^{-1})^{-r}$ . The  $x_i^\pm, k_i^{\pm 1}$  are called *Chevalley generators*.

We use the coproduct  $\Delta : \mathcal{U}_q(\mathfrak{g}) \rightarrow \mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g})$  defined for  $0 \leq i \leq n$  by

$$\Delta(k_i) = k_i \otimes k_i, \Delta(x_i^+) = x_i^+ \otimes 1 + k_i \otimes x_i^+, \Delta(x_i^-) = x_i^- \otimes k_i^{-1} + 1 \otimes x_i^-.$$

This is the same choice as in [D1, C, FM]<sup>1</sup>.

**2.2. Drinfeld generators.** The indecomposable affine Cartan matrices are classified [Kac] into two main classes, *twisted* types and *untwisted* types. The latest includes *simply-laced* types and *untwisted non simply-laced* types. The type of  $C$  is denoted by  $X$ . We use the numbering of nodes as in [Kac] if  $X \neq A_{2n}^{(2)}$ , and we use the reversed numbering if  $X = A_{2n}^{(2)}$ . Let  $r$  be the *twisting order* of  $\mathfrak{g}$ , that is  $r = 1$  is the untwisted cases, otherwise  $r = 2$  if  $X \neq D_4^{(3)}$  and  $r = 3$  if  $X = D_4^{(3)}$ .

We set  $\mu_i = 1$  for  $0 \leq i \leq n$ , except when  $(X, i) = (A_{2n}^{(2)}, n)$  where we set  $\mu_n = 2$ . Without loss of generality, we can choose the  $r_i$  so that  $\mu_i r_i \in \mathbb{N}^*$  for any  $i$  and  $(\mu_0 r_0 \wedge \cdots \wedge \mu_n r_n) = 1$  (there is a unique such choice). Let  $i \in I = \{1, \dots, n\}$ . If  $r = r_i > 1$  we set  $d_i = r_i$ . We set  $d_i = 1$  otherwise. So for the twisted types we have  $d_i = \mu_i r_i$ , and for the untwisted types we have  $d_i = 1$ .

Let  $\bar{\mathfrak{g}}$  be the finite-dimensional simple Lie algebra of Cartan matrix  $(C_{i,j})_{i,j \in I}$ . We denote respectively by  $\omega_i, \alpha_i, \alpha_i^\vee$  ( $i \in I$ ) the fundamental weights, the simple roots and the simple coroots of  $\bar{\mathfrak{g}}$ . We use the standard partial ordering  $\leq$  on the weight lattice  $P$  of  $\bar{\mathfrak{g}}$ .

The algebra  $\mathcal{U}_q(\mathfrak{g})$  has another set of generators, called *Drinfeld generators*, denoted by

$$x_{i,m}^\pm, k_i^{\pm 1}, h_{i,r}, c^{\pm 1/2} \text{ for } i \in I, m \in \mathbb{Z}, r \in \mathbb{Z} \setminus \{0\},$$

and defined from the Chevalley generators by using the action of Lusztig automorphisms of  $\mathcal{U}_q(\mathfrak{g})$  (in [B] for the untwisted types and in [D1] for the twisted types). We have  $x_i^\pm = x_{i,0}^\pm$  for  $i \in I$ . A complete set of relations for Drinfeld relations was obtained in [B, BCP, D2]. In particular the multiplication defines a surjective linear morphism

$$(1) \quad \mathcal{U}_q^-(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{h}) \otimes \mathcal{U}_q^+(\mathfrak{g}) \rightarrow \mathcal{U}_q(\mathfrak{g})$$

where  $\mathcal{U}_q^\pm(\mathfrak{g})$  is the subalgebra generated by the  $x_{i,m}^\pm$  ( $i \in I, m \in \mathbb{Z}$ ) and  $\mathcal{U}_q(\mathfrak{h})$  is the subalgebra generated by the  $k_i^{\pm 1}$ , the  $h_{i,r}$  and  $c^{\pm 1/2}$  ( $i \in I, r \in \mathbb{Z} \setminus \{0\}$ ).

<sup>1</sup>In [Kas] another coproduct is used. We recover the coproduct used in the present paper by taking the opposite coproduct and changing  $q$  to  $q^{-1}$ .

**2.3. Finite-dimensional representations.** For  $i \in I$ , the action of  $k_i$  on any object of  $\mathcal{F}$  is diagonalizable with eigenvalues in  $\pm q^{r_i \mathbb{Z}}$ . Without loss of generality, we can assume that  $\mathcal{F}$  is the category of *type 1* finite-dimensional representations (see [CP2]), i.e. we assume that for any object of  $\mathcal{F}$ , the eigenvalues of  $k_i$  are in  $q^{r_i \mathbb{Z}}$  for  $i \in I$ .

The simple objects of  $\mathcal{F}$  have been classified by Chari-Pressley [CP1, CP2, CP3] (see [H2] for some complements in the twisted cases). The simple objects are parametrized by  $n$ -tuples of polynomials  $(P_i(u))_{i \in I}$  satisfying  $P_i(0) = 1$  (they are called *Drinfeld polynomials*).

For  $\omega \in P$ , the *weight space*  $V_\omega$  of an object  $V$  in  $\mathcal{F}$  is the set of *weight vectors* of weight  $\omega$ , i.e. of vectors  $v \in V$  satisfying  $k_i v = q^{(r_i \omega(\alpha_i^\vee))} v$  for any  $i \in I$ . Thus, we have the weight decomposition

$$V = \bigoplus_{\omega \in P} V_\omega.$$

Let us define the  $\ell$ -weight weight decomposition which is its refinement.

The elements  $c^{\pm 1/2}$  acts by identity on any object  $V$  of  $\mathcal{F}$ , and so the action of the  $h_{i,r}$  commute. Since the  $h_{i,r}$ ,  $i \in I$ ,  $r \in \mathbb{Z} \setminus \{0\}$ , also commute with the  $k_i$ ,  $i \in I$ , every object in  $\mathcal{F}$  can be decomposed as a direct sum of generalized eigenspaces of the  $h_{i,r}$  and  $k_i$ . More precisely, by Frenkel-Reshetikhin theory of  $q$ -characters [FR], the eigenvalues of the  $h_{i,r}$  and  $k_i$  can be *encoded* by *monomials*  $m$  in formal variables  $Y_{i,a}^{\pm 1}$  ( $i \in I, a \in \mathbb{C}^*$ ). The construction<sup>2</sup> is extended to twisted types in [H2]. Let  $\mathcal{M}$  be the set of such monomials (also called  *$l$ -weights*). Given  $m \in \mathcal{M}$  and an object  $V$  in  $\mathcal{F}$ , let  $V_m$  be the corresponding generalized eigenspace of  $V$  (also called  *$l$ -weight spaces*); thus,

$$V = \bigoplus_{m \in \mathcal{M}} V_m.$$

As mentioned above, the decomposition in  $l$ -weight spaces is finer than the decomposition in weight spaces. Indeed, if  $v \in V_m$ , then  $v$  is a weight vector of weight

$$\omega(m) = \sum_{i \in I, a \in \mathbb{C}^*} u_{i,a}(m) \mu_i \omega_i \in P,$$

where we denote  $m = \prod_{i \in I, a \in \mathbb{C}^*} Y_{i,a}^{u_{i,a}(m)}$ . For  $v \in V_m$ , we set  $\omega(v) = \omega(m)$ .

The  $q$ -character morphism is an injective ring morphism

$$\chi_q : \text{Rep}(\mathcal{U}_q(\mathfrak{g})) \rightarrow \mathcal{Y} = \mathbb{Z} \left[ Y_{i,a}^{\pm 1} \right]_{i \in I, a \in \mathbb{C}^*},$$

$$\chi_q(V) = \sum_{m \in \mathcal{M}} \dim(V_m) m.$$

**Remark 2.1.** For any  $i \in I$ ,  $r \in \mathbb{Z} \setminus \{0\}$ ,  $m, m' \in \mathcal{M}$ , the eigenvalue of  $h_{i,r}$  associated to  $mm'$  is the sum of the eigenvalues of  $h_{i,r}$  associated respectively to  $m$  and  $m'$  [FR, H2].

<sup>2</sup>For the twisted types there is a modification of the theory and we consider two kinds of variables in [H2]. For homogeneity of notations, the  $Y_{i,a}$  in the present paper are the  $Z_{i,a}$  of [H2]. We do not use in this paper the variables denoted by  $Y_{i,a}$  in [H2].

If  $V_m \neq \{0\}$  we say that  $m$  is an  $l$ -weight of  $V$ . A vector  $v$  belonging to an  $l$ -weight space  $V_m$  is called an  $l$ -weight vector. We denote  $M(v) = m$  the  $l$ -weight of  $v$ . A *highest  $l$ -weight vector* is an  $l$ -weight vector  $v$  satisfying  $x_{i,p}^+ v = 0$  for any  $i \in I, p \in \mathbb{Z}$ .

A monomial  $m \in \mathcal{M}$  is said to be *dominant* if  $u_{i,a}(m) \geq 0$  for any  $i \in I, a \in \mathbb{C}^*$ . For  $V$  a simple object in  $\mathcal{F}$ , let  $M(V)$  be the *highest weight monomial* of  $\chi_q(V)$ , that is so that  $\omega(M(V))$  is maximal for the partial ordering on  $P$ .  $M(V)$  is dominant and characterizes the isomorphism class of  $V$  (it is equivalent to the data of the Drinfeld polynomials). Hence to a dominant monomial  $M$  is associated a simple representation  $L(M)$ . For  $i \in I$  and  $a \in \mathbb{C}^*$ , we define the fundamental representation

$$V_i(a) = L\left(Y_{i,a^{d_i}}\right).$$

**Example 2.2.** *The  $q$ -character of the fundamental representation  $L(Y_a)$  of  $\mathcal{U}_q(\hat{\mathfrak{sl}}_2)$  is*

$$\chi_q(L(Y_a)) = Y_a + Y_{aq^2}^{-1}.$$

Let  $i \in I, a \in \mathbb{C}^*$  and let us define the monomial  $A_{i,a}$  analog of a simple root. For the untwisted cases, we set [FR]

$$\begin{aligned} A_{i,a} &= Y_{i,aq^{-r_i}} Y_{i,aq^{r_i}} \times \left( \prod_{\{j \in I | C_{i,j} = -1\}} Y_{j,a} \right)^{-1} \\ &\times \left( \prod_{\{j \in I | C_{i,j} = -2\}} Y_{j,aq^{-1}} Y_{j,aq} \right)^{-1} \times \left( \prod_{\{j \in I | C_{i,j} = -3\}} Y_{j,aq^{-2}} Y_{j,a} Y_{j,aq^2} \right)^{-1}. \end{aligned}$$

Recall  $r$  the twisting number defined above. Let  $\epsilon$  be a primitive  $r$ th root of 1. We now define  $A_{i,a}$  for the twisted types as in [H2].

If  $(X, i) \neq (A_{2n}^{(2)}, n)$  and  $r_i = 1$ , we set

$$A_{i,a} = Y_{i,aq^{-1}} Y_{i,aq} \times \left( \prod_{\{j \in I | C_{i,j} < 0\}} Y_{j,a^{(r_j C_{j,i})}} \right)^{-1}.$$

If  $(X, i) \neq (A_{2n}^{(2)}, n)$  and  $r_i > 1$ , we set

$$A_{i,a} = Y_{i,aq^{-r_i}} Y_{i,aq^{r_i}} \times \left( \prod_{\{j \in I | C_{i,j} < 0, r_j = r\}} Y_{j,a} \right)^{-1} \times \left( \prod_{\{j \in I | C_{i,j} < 0, r_j = 1\}} \left( \prod_{\{b \in \mathbb{C}^* | (b)^r = a\}} Y_{j,b} \right) \right)^{-1}.$$

If  $(X, i) = (A_{2n}^{(2)}, n)$ , we set  $A_{n,a} = \begin{cases} Y_{n,aq^{-1}} Y_{n,aq} Y_{n,-a}^{-1} Y_{n-1,a}^{-1} & \text{if } n > 1, \\ Y_{1,aq^{-1}} Y_{1,aq} Y_{1,-a}^{-1} & \text{if } n = 1. \end{cases}$

For  $i \in I, a \in \mathbb{C}^*$ , we have [FM, H2] :

$$(2) \quad \chi_q(V_i(a)) \in Y_{i,a^{d_i}} + Y_{i,a^{d_i}} A_{i,(a^{d_i} q^{\mu_i r_i})}^{-1} \mathbb{Z} \left[ A_{j,(a^{\epsilon^k q^r})^{d_j}}^{-1} \right]_{j \in I, k \in \mathbb{Z}, r > 0}.$$

As a simple module  $L(m)$  is a subquotient of a tensor product of fundamental representations, this implies

$$\chi_q(L(m)) \in m\mathbb{Z}[A_{i,a}^{-1}]_{i \in I, a \in \mathbb{C}^*}.$$

**2.4. Properties of  $l$ -weight vectors.** Let  $\mathcal{U}_q(\mathfrak{h})^+$  be the subalgebra of  $\mathcal{U}_q(\mathfrak{h})$  generated by the  $k_i^{\pm 1}$  and the  $h_{i,r}$  ( $i \in I, r > 0$ ). The  $q$ -character and the decomposition in  $l$ -weight spaces of a representation in  $\mathcal{F}$  is completely determined by the action of  $\mathcal{U}_q(\mathfrak{h})^+$  [FR, H2]. Therefore one can define the  $q$ -character  $\chi_q(W)$  of a  $\mathcal{U}_q(\mathfrak{h})^+$ -submodule  $W$  of an object in  $\mathcal{F}$ .

The following result describes a condition on the  $l$ -weight of a linear combination of pure tensor products of weight vectors.

**Theorem 2.3.** [H3] *Let  $V_1, V_2$  in  $\mathcal{F}$  and consider an  $l$ -weight vector*

$$w = \left( \sum_{\alpha} w_{\alpha} \otimes v_{\alpha} \right) + \left( \sum_{\beta} w'_{\beta} \otimes v'_{\beta} \right) \in V_1 \otimes V_2$$

*satisfying the following conditions.*

- (i) *The  $v_{\alpha}$  are  $l$ -weight vectors and the  $v'_{\beta}$  are weight vectors.*
- (ii) *For any  $\beta$ , there is an  $\alpha$  satisfying  $\omega(v'_{\beta}) > \omega(v_{\alpha})$ .*
- (iii) *For  $\omega \in \{\omega(v_{\alpha})\}_{\alpha}$ , we have  $\sum_{\{\alpha | \omega(v_{\alpha}) = \omega\}} w_{\alpha} \otimes v_{\alpha} \neq 0$ .*

*Then  $M(w)$  is the product of one  $M(v_{\alpha})$  by an  $l$ -weight of  $V_1$ .*

**2.5. Cyclicity and intertwiners.** We have the following cyclicity result [C, Kas, VV] discussed in the in the introduction. We set  $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ .

**Theorem 2.4.** *Suppose that  $a_1, \dots, a_R \in \mathbb{C}^*$  satisfy  $a_p a_s^{-1} \notin \epsilon^{\mathbb{Z}} q^{-\mathbb{N}^*}$  for all  $1 \leq s < p \leq R$ . Then for any  $i_1, \dots, i_R \in I$  the module*

$$V_{i_R}(a_R) \otimes \dots \otimes V_{i_1}(a_1)$$

*is cyclic. Moreover, there exists a unique, up to a scalar, non-zero morphism*

$$V_{i_R}(a_R) \otimes \dots \otimes V_{i_1}(a_1) \rightarrow V_{i_1}(a_1) \otimes \dots \otimes V_{i_R}(a_R).$$

*Its image is simple isomorphic to  $L\left(Y_{i_1, (a_1)^{d_{i_1}}} \cdots Y_{i_R, (a_R)^{d_{i_R}}}\right)$ .*

Note that an arbitrary dominant monomial  $m$  can be factorized as a product

$$m = Y_{i_1, a_1^{d_1}} \cdots Y_{i_R, a_R^{d_R}}$$

so that the ordered sequence of fundamental representations  $V_{i_j}(a_j)$  satisfy the hypothesis of Theorem 2.4. Hence it implies the following<sup>3</sup>.

**Proposition 2.5.** *A cyclic tensor product of simple representations is isomorphic to a quotient of a cyclic tensor product of fundamental representations.*

<sup>3</sup>The statement of Proposition 2.5 is also true for an arbitrary cyclic module [BN], but will not be used in such a generality in the present paper.

Let  $W = L(m)$  and  $W' = L(m')$  simple representations.

As a direct consequence of Theorem 2.4, we get the following (see [H3] for more comments).

**Corollary 2.6.** *Suppose that  $m, m' \in \mathcal{M}$  are such that  $u_{i, a^{d_i}}(m) \neq 0$ ,  $i \in I$ ,  $a \in \mathbb{C}^*$  implies that  $u_{j, (a q^s \epsilon^k)^{d_j}}(m') = 0$  for all  $j \in I$ ,  $s > 0$ ,  $k \in \mathbb{Z}$ . Then  $W \otimes W'$  is cyclic and there exists a non-zero morphism of  $\mathcal{U}_q(\mathfrak{g})$ -modules, unique up to a constant multiple*

$$\mathcal{I}_{W, W'} : W \otimes W' \rightarrow W' \otimes W.$$

Its image is simple isomorphic to  $L(mm')$ .

For  $a \in \mathbb{C}^*$  generic, we have an isomorphism of  $\mathcal{U}_q(\mathfrak{g})$ -modules

$$\mathcal{T}_{W, W'}(a) : W \otimes W'(a) \rightarrow W'(a) \otimes W.$$

Here  $a$  generic means that  $a$  is in the complement of a finite set of  $\mathbb{C}^*$  (which depends on  $W$  and  $W'$ ) and  $W'(a)$  is the twist of  $W'$  by the algebra automorphism  $\tau_a$  of  $\mathcal{U}_q(\mathfrak{g})$  defined [CP1] on the Drinfeld generators by

$$\tau_a \left( x_{i, m}^{\pm} \right) = a^{\pm m} x_{i, m}^{\pm}, \quad \tau_a \left( h_{i, r} \right) = a^r h_{i, r}, \quad \tau_a \left( k_i^{\pm 1} \right) = k_i^{\pm 1}, \quad \tau_a \left( c^{\pm \frac{1}{2}} \right) = c^{\pm \frac{1}{2}}.$$

Considering  $a$  as a variable  $z$ , we get a rational map in  $z$

$$\mathcal{T}_{W, W'}(z) : (W \otimes W') \otimes \mathbb{C}(z) \rightarrow (W' \otimes W) \otimes \mathbb{C}(z).$$

This map is normalized so that for  $v \in W$ ,  $v' \in W'$  highest weight vectors, we have

$$(\mathcal{T}_{W, W'}(z))(v \otimes v') = v' \otimes v.$$

The map  $\mathcal{T}_{W, W'}(z)$  is invertible and

$$(3) \quad (\mathcal{T}_{W, W'}(z))^{-1} = \mathcal{T}_{W', W}(z^{-1}).$$

If  $W \otimes W'$  is cyclic,  $\mathcal{T}_{W, W'}$  has a limit at  $z = 1$  which is denoted by  $\mathcal{I}_{W, W'}$  as above (and which is not invertible in general).

Note that  $\mathcal{T}_{W, W'}(z)$  defines a morphism of representation of

$$U_{q, z}(\mathfrak{g}) = \mathcal{U}_q(\mathfrak{g}) \otimes \mathbb{C}(z)$$

if the action on  $W'$  is twisted by the automorphism  $\tau_z$  of  $U_{q, z}(\mathfrak{g})$  defined as  $\tau_a$  with  $a$  replaced by the formal variable  $z$ . The corresponding  $U_{q, z}(\mathfrak{g})$ -module  $W' \otimes \mathbb{C}(z)$  is denoted by  $W'(z)$ . See [H4] for references.

**Remark 2.7.** *By results of Drinfeld, it is well-known (see e.g. [CP2, Section 12.5.B]) that the intertwiners  $\mathcal{T}_{W, W'}(z)$  can be obtained from the universal  $R$ -matrix of  $\mathcal{U}_q(\mathfrak{g})$  which is a solution of the Yang-Baxter equation. This implies the hexagonal relation :*

for  $U, V, W$  representations as above we have a commutative diagram :

$$(4) \quad \begin{array}{ccc} & V \otimes U \otimes W \xrightarrow{Id \otimes \mathcal{T}_{U,W}(zw)} & V \otimes W \otimes U \\ \mathcal{T}_{U,V}(z) \otimes Id \nearrow & & \searrow \mathcal{T}_{V,W}(w) \otimes Id \\ U \otimes V \otimes W & & W \otimes V \otimes U \\ Id \otimes \mathcal{T}_{V,W}(w) \searrow & & \nearrow Id \otimes \mathcal{T}_{U,V}(z) \\ & U \otimes W \otimes V \xrightarrow{\mathcal{T}_{U,W}(zw) \otimes Id} & W \otimes U \otimes V \end{array}$$

Here the tensor products are taken over  $\mathbb{C}(z, w)$  (for clarity we have omitted the variables  $z, w$  in the diagram).

**Remark 2.8.** If  $W \otimes W'$  is cyclic, the morphism  $\mathcal{I}_{W,W'}$  has also the naive deformation

$$\mathcal{I}_{W,W'}(z) : (W \otimes W') \otimes \mathbb{C}(z) \rightarrow (W' \otimes W) \otimes \mathbb{C}(z)$$

obtained by extension of scalars from  $\mathbb{C}$  to  $\mathbb{C}(z)$ . It is a morphism of  $U_{q,z}(\mathfrak{g})$ -modules (the action on  $W$  and  $W'$  are not twisted). It is an isomorphism if and only if  $\mathcal{I}_{W,W'}$  is an isomorphism.

**Remark 2.9.** Let

$$\mathcal{A} \subset \mathbb{C}(z)$$

be the ring of rational functions without pole at  $z = 1$  and

$$\mathcal{U}_{q,\mathcal{A}} = \mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{A}.$$

For  $S$  simple in  $\mathcal{F}$ , we have the  $\mathcal{U}_{q,\mathcal{A}}(\mathfrak{g})$ -modules  $S_{\mathcal{A}} = S \otimes \mathcal{A}$  and

$$S_{\mathcal{A}}(z) = \mathcal{U}_{q,\mathcal{A}}(\mathfrak{g}) \cdot v \subset S \otimes \mathbb{C}(z)$$

where  $v$  is a highest weight vector of  $S$  and the action is twisted by  $\tau_z$  in  $S \otimes \mathbb{C}(z)$ . This is a  $\mathcal{A}$ -module of rank  $\dim(S)$ , that is an  $\mathcal{A}$ -lattice in  $S \otimes \mathbb{C}(z)$ . Suppose that  $W \otimes W'$  is cyclic. Then the morphism  $\mathcal{T}_{W,W'}(z)$ ,  $\mathcal{I}_{W,W'}(z)$  make sense as morphisms of  $\mathcal{U}_{q,\mathcal{A}}(\mathfrak{g})$ -modules between the  $\mathcal{A}$ -lattices :

$$\mathcal{T}_{W,W'}(z) : W_{\mathcal{A}} \otimes_{\mathcal{A}} W'_{\mathcal{A}}(z) \rightarrow W'_{\mathcal{A}}(z) \otimes_{\mathcal{A}} W_{\mathcal{A}}.$$

$$\mathcal{I}_{W,W'}(z) : W_{\mathcal{A}} \otimes_{\mathcal{A}} W'_{\mathcal{A}} \rightarrow W'_{\mathcal{A}} \otimes_{\mathcal{A}} W_{\mathcal{A}}.$$

This clear for  $\mathcal{I}_{W,W'}(z)$ . For  $\mathcal{T}_{W,W'}(z)$ , let  $v \in W_{\mathcal{A}} \otimes_{\mathcal{A}} W'_{\mathcal{A}}(z)$  highest weight. We may assume that

$$\mathcal{T}_{W,W'}(z)(v) \in W'_{\mathcal{A}} \otimes_{\mathcal{A}} W_{\mathcal{A}}.$$

As  $W \otimes W'$  is cyclic, it suffices to check that for  $g \in \mathcal{U}_q(\mathfrak{g})$ ,

$$\mathcal{T}_{W,W'}(z)(g.v) \in W'_{\mathcal{A}}(z) \otimes_{\mathcal{A}} W_{\mathcal{A}}.$$

This is clear as  $\mathcal{T}_{W,W'}(z)$  is a morphism.

### 3. SUBCATEGORIES AND REDUCTIONS

In Section 3 we reduce the problem by considering certain remarkable subcategories  $\mathcal{C}_{\mathbb{Z}}, \mathcal{C}_{\ell}$  which have a compatibility property with  $q$ -characters (Theorem 3.5).

### 3.1. The category $\mathcal{C}_{\mathbb{Z}}$ .

**Definition 3.1.** *The category  $\mathcal{C}_{\mathbb{Z}}$  is the full subcategory of objects in  $\mathcal{F}$  whose Jordan-Hölder composition factors are simple representations  $V$  satisfying*

$$M(V) \in \mathbb{Z} \left[ Y^{\pm 1} \right]_{i \in I, l, k \in \mathbb{Z}}^{i, (q^l \epsilon^k)^{d_i}}.$$

The category  $\mathcal{C}_{\mathbb{Z}}$  is particularly important as the description of the simple objects of  $\mathcal{F}$  essentially reduces to the description of the simple objects of  $\mathcal{C}_{\mathbb{Z}}$ , see [HL1, Section 3.7].

**Proposition 3.2.** *It suffices to prove the statement of Theorem 1.1 for simple representations  $S_i$  in the category  $\mathcal{C}_{\mathbb{Z}}$ .*

*Proof.* Let  $S = L(M)$  be a simple representation. Then there is a unique factorization<sup>4</sup>

$$M = \prod_{a \in \mathbb{C}^*/(q^{\mathbb{Z}}\epsilon^{\mathbb{Z}})} M_a$$

where

$$M_a \in \mathbb{Z} \left[ Y^{\pm 1} \right]_{i \in I, l, k \in \mathbb{Z}}^{i, (aq^l \epsilon^k)^{d_i}}.$$

Then  $S$  has a factorization into simple modules

$$S \simeq \bigotimes_{a \in \mathbb{C}^*/(q^{\mathbb{Z}}\epsilon^{\mathbb{Z}})} (S)_a$$

where

$$S_a = L(M_a).$$

Indeed the irreducibility of this tensor product follows from Corollary 2.6 and from the fact that a cyclic module whose dual is cyclic is simple (see [CP1, Section 4.10]).

Now it suffices to prove that for  $S_1, \dots, S_N$  simple modules, the module

$$S_1 \otimes \dots \otimes S_N$$

is cyclic if and only if

$$(S_1)_a \otimes \dots \otimes (S_N)_a$$

is cyclic for any  $a \in \mathbb{C}^*/(q^{\mathbb{Z}}\epsilon^{\mathbb{Z}})$ .

Note that for any simple module  $S, S'$  and for any  $a \neq b \in \mathbb{C}^*/(q^{\mathbb{Z}}\epsilon^{\mathbb{Z}})$ , we have a morphism

$$I_{(S)_a, (S')_b} : (S)_a \otimes (S')_b \simeq (S')_b \otimes (S)_a.$$

It is an isomorphism as  $I_{(S')_b, S_a}$  is also well-defined. Hence

$$S_1 \otimes \dots \otimes S_N \simeq \bigotimes_{a \in \mathbb{C}^*/(q^{\mathbb{Z}}\epsilon^{\mathbb{Z}})} (S_1)_a \otimes \dots \otimes (S_N)_a.$$

---

<sup>4</sup>This factorization might be seen as an analog of the Steinberg theorem in modular representation theory.

In particular, the "only if" part is clear. Conversely, by Proposition 2.5, for each  $a \in \mathbb{C}^*/(q^{\mathbb{Z}}\epsilon^{\mathbb{Z}})$  there is a cyclic tensor product  $W_a$  of fundamental representations with a morphism

$$\phi_a : W_a \rightarrow (S_1)_a \otimes \cdots \otimes (S_N)_a$$

which is surjective by hypothesis. But

$$W = \bigotimes_{a \in \mathbb{C}^*/(q^{\mathbb{Z}}\epsilon^{\mathbb{Z}})} W_a$$

is cyclic by Theorem 2.4 and

$$\bigotimes_{a \in \mathbb{C}^*/(q^{\mathbb{Z}}\epsilon^{\mathbb{Z}})} \phi_a : W \rightarrow \bigotimes_{a \in \mathbb{C}^*/(q^{\mathbb{Z}}\epsilon^{\mathbb{Z}})} (S_1)_a \otimes \cdots \otimes (S_N)_a \simeq S_1 \otimes \cdots \otimes S_N$$

is surjective. Hence the result.  $\square$

### 3.2. The categories $\mathcal{C}_\ell$ .

**Definition 3.3.** Given a non-negative integer  $\ell$ ,  $\mathcal{C}_\ell$  is the full subcategory of  $\mathcal{C}_{\mathbb{Z}}$  whose Jordan-Hölder composition factors are simple representations  $V$  satisfying

$$M(V) \in \mathcal{Y}_1 = \mathbb{Z} \left[ Y_{i, (q^l \epsilon^k)^{d_i}} \right]_{i \in I, 0 \leq l \leq \ell, k \in \mathbb{Z}}.$$

The categories  $\mathcal{C}_\ell$  are stable under tensor product (see [H3, Lemma 4.10]). Up to a twist by the automorphism  $\tau_a$  for a certain  $a \in \mathbb{C}^*$ , a family of simple modules in  $\mathcal{C}_{\mathbb{Z}}$  is in a subcategory  $\mathcal{C}_\ell$  (see [HL1] and [H3, Section 4.2]). Hence as in [H3, Section 4.2], we see that it suffices to prove the statement for the categories  $\mathcal{C}_\ell$ .

The statement of Theorem 1.1 is clear for the category  $\mathcal{C}_0$  as all simple objects of  $\mathcal{C}_0$  are tensor products of fundamental representations in  $\mathcal{C}_0$ . Moreover an arbitrary tensor product of simple objects in  $\mathcal{C}_0$  is simple (see [H3, Lemma 4.11]).

**3.3. Upper  $q$ -characters.** Let us remind a technical result about the "upper" part of the  $q$ -character of a simple module.

Fix  $L \in \mathbb{Z}$ . For a dominant monomial  $m \in \mathcal{Y}_1$ , we denote by  $m^{\leq L}$  the product with multiplicities of the factors  $Y_{i, (\epsilon^k q^l)^{d_i}}^{\pm 1}$  occurring in  $m$  with  $l \leq L$ ,  $i \in I$ ,  $k \in \mathbb{Z}$ . We also set

$$m = m^{\geq L} m^{\leq L-1}.$$

**Definition 3.4.** The upper  $q$ -character  $\chi_{q, \geq L}(V)$  is the sum with multiplicities of the monomials  $m$  occurring in  $\chi_q(V)$  satisfying

$$m^{\leq (L-1)} = M^{\leq (L-1)}.$$

The following technical result will be useful in the following.

**Theorem 3.5.** [H3] For  $M \in \mathcal{Y}_1$  a dominant monomial and  $L \in \mathbb{Z}$ , we have

$$\chi_{q, \geq L}(L(M)) = M^{\leq (L-1)} \chi_q(L(M^{\geq L})).$$

## 4. END OF THE PROOF OF THEOREM 1.1

In this section we finish the proof of Theorem 1.1. We first prove a preliminary result (Proposition 4.1) asserting that the cyclicity of a tensor product  $S \otimes S'$  implies the cyclicity of a certain submodule  $S_+ \otimes S'_+$ . Then we prove the result by using an induction on  $N$  and on the size of a subcategory  $\mathcal{C}_\ell$ : we prove the cyclicity of an intermediate module in Lemma 4.2 and the final arguments make intensive use of intertwiners and of their deformations derived from  $R$ -matrices.

**4.1. Preliminary result.** Let  $S$  be a simple module in  $\mathcal{C}_\ell$  of highest weight monomial  $M$ . Let

$$M_- = M^{\leq 0} \text{ and } M_+ = M^{\geq 1} \text{ so that } M = M_+ M_-.$$

Set

$$S_\pm = L(M_\pm).$$

Recall the surjective morphism of Corollary 2.6.

$$\mathcal{I}_{S_+, S_-} : S_+ \otimes S_- \rightarrow S \subset S_- \otimes S_+.$$

**Proposition 4.1.** *Let  $S, S'$  simple objects in  $\mathcal{C}_\ell$  such that the tensor product  $S \otimes S'$  is cyclic. Then the tensor product  $S_+ \otimes S'_+$  is cyclic.*

*Proof.* Consider the morphism

$$(\mathcal{I}_{S_+, S_-} \otimes \mathcal{I}_{S'_+, S'_-}) \circ (\text{Id} \otimes \mathcal{I}_{S'_+, S'_-} \otimes \text{Id}) :$$

$$S_+ \otimes S'_+ \otimes S_- \otimes S'_- \rightarrow S_+ \otimes S_- \otimes S'_+ \otimes S'_- \rightarrow S \otimes S'.$$

Then, as  $S \otimes S'$  is cyclic, it is isomorphic to a quotient of the submodule

$$W \subset S_+ \otimes S'_+ \otimes S_- \otimes S'_-$$

generated by an highest weight vector. Hence monomials have higher multiplicities in  $\chi_q^{\geq 1}(W)$  than in

$$\chi_q^{\geq 1}(S \otimes S') = \chi_q^{\geq 1}(S) \chi_q^{\geq 1}(S') = \chi_q(S_+ \otimes S'_+) M_- M'_-.$$

By Formula (2) and Theorem 3.5, we have :

$$\chi_q^{\geq 1}(S_-) = M_- \text{ and } \chi_q^{\geq 1}(S_+) = \chi_q(S_+) = (M_-)^{-1} \chi_q^{\geq 1}(S).$$

Consider an  $\ell$ -weight vector

$$w \in S_+ \otimes S'_+ \otimes S_- \otimes S'_-$$

whose  $\ell$ -weight  $\gamma$  contributes to

$$\chi_q(S_+ \otimes S'_+) M_- M'_-.$$

Then by Theorem 2.3 we have

$$w \in S_+ \otimes S'_+ \otimes v$$

where  $v$  is a highest weight vector of  $S_- \otimes S'_-$ . Otherwise  $\gamma$  would be a product

$$\gamma = \gamma_1 \times \gamma_2$$

of an  $\ell$ -weight  $\gamma_1$  of  $S_+ \otimes S'_+$  by an  $\ell$ -weight  $\gamma_2$  of  $S_- \otimes S'_-$  satisfying  $\gamma_2 \neq M_- M'_-$ . Consider the factorization of  $M_- M'_- \gamma_2^{-1}$  as a product of  $A_{i,a}$  as explained at the end

of section 2.3. Note that the  $A_{i,a}$  are algebraically independent [FR, H2]. Then by Formula (2) at least one factor  $A_{i,a}$  would occur with

$$i \in I \text{ and } a \notin \epsilon^{d_i \mathbb{Z}} q^{d_i(1+\mathbb{N}) + \mu_i r_i}.$$

This is a contradiction with Theorem 3.5 as  $\gamma$  occurs in

$$\chi_q(S_+ \otimes S'_+) M_- M'_-.$$

We have proved

$$S_+ \otimes S'_+ \otimes v \subset W.$$

This implies the result.  $\square$

**4.2. Final arguments.** We prove Theorem 1.1 by induction on  $\ell$  and on  $N$ . As explained above in section 3.2, we have checked the result for  $\ell = 0$ . Moreover the result is trivial for  $N = 2$ .

Let  $S_1, \dots, S_N$  as in the statement of Theorem 1.1. First let us prove the following.

**Lemma 4.2.** *The module*

$$V = S_{1,+} \otimes S_2 \otimes \dots \otimes S_N \otimes S_{1,-}$$

*is cyclic.*

*Proof.* By Proposition 4.1, the induction hypothesis on  $N$  implies that the modules

$$S_2 \otimes \dots \otimes S_N \text{ and } S_{2,+} \otimes \dots \otimes S_{N,+}$$

are cyclic. Besides, it follows from Corollary 2.6 that for  $i \neq j$ ,

$$S_{i,+} \otimes S_{j,-}$$

is cyclic. Composing applications

$$\text{Id} \otimes \mathcal{I}_{S_{i,+}, S_{j,-}} \otimes \text{Id}$$

for  $i \neq j$ , we get a surjective morphism

$$\phi : (S_{2,+} \otimes \dots \otimes S_{N,+}) \otimes (S_{2,-} \otimes \dots \otimes S_{N,-}) \rightarrow S_2 \otimes \dots \otimes S_N.$$

Hence the map

$$\text{Id} \otimes \phi \otimes \text{Id}$$

$$V' = S_{1,+} \otimes (S_{2,+} \otimes \dots \otimes S_{N,+} \otimes S_{2,-} \otimes \dots \otimes S_{N,-}) \otimes S_{1,-} \rightarrow V$$

is surjective.

By Proposition 4.1, the induction hypothesis on  $\ell$  implies that the module

$$S_{1,+} \otimes S_{2,+} \otimes \dots \otimes S_{N,+}$$

is cyclic. By Proposition 2.5, it is isomorphic to a quotient of a cyclic tensor product  $W$  of fundamental representations. Moreover

$$S_{2,-} \otimes \dots \otimes S_{N,-}$$

is a simple tensor product of fundamental representations. Then

$$W \otimes S_{2,-} \otimes \dots \otimes S_{N,-}$$

is a cyclic tensor product of fundamental representation which admits  $V'$  as a quotient. Hence  $V'$  is cyclic and  $V$  is cyclic.  $\square$

Now let us end the proof of Theorem 1.1.

We have a sequence of maps

$$\begin{aligned} & (\mathcal{I}_{S_{1,+}, S_{1,-}} \otimes \text{Id}) \circ \cdots \circ (\text{Id} \otimes \mathcal{I}_{S_{N-1}, S_{1,-}} \otimes \text{Id}) \circ (\text{Id} \otimes \mathcal{I}_{S_N, S_{1,-}}) \\ V \rightarrow & S_{1,+} \otimes S_2 \otimes \cdots \otimes S_{N-1} \otimes S_{1,-} \otimes S_N \rightarrow S_{1,+} \otimes S_2 \otimes \cdots \otimes S_{N-2} \otimes S_{1,-} \otimes S_{N-1} \otimes S_N \\ & \rightarrow \cdots \rightarrow S_{1,+} \otimes S_{1,-} \otimes S_2 \otimes \cdots \otimes S_N \rightarrow S_1 \otimes \cdots \otimes S_N. \end{aligned}$$

Let us replace each module

$$S_{1,+} \otimes S_2 \otimes \cdots \otimes S_i \otimes S_{1,-} \otimes S_{i+1} \otimes \cdots \otimes S_N$$

by its quotient

$$(5) \quad \overline{S_{1,+} \otimes S_2 \otimes \cdots \otimes S_i \otimes S_{1,-} \otimes S_{i+1} \otimes \cdots \otimes S_N} = S_{1,+} \otimes S_2 \otimes \cdots \otimes S_i \otimes S_{1,-} \otimes S_{i+1} \otimes \cdots \otimes S_N / \text{Ker}$$

by the kernel<sup>5</sup> Ker of the morphism

$$(\mathcal{I}_{S_{1,+}, S_{1,-}} \otimes \text{Id}) \circ \cdots \circ (\text{Id} \otimes \mathcal{I}_{S_i, S_{1,-}} \otimes \text{Id})$$

to  $S_1 \otimes \cdots \otimes S_N$ . In the following we call such a quotient a bar-module.

We get a composition

$$\mathcal{I}_2 \circ \mathcal{I}_3 \circ \cdots \circ \mathcal{I}_N$$

of well-defined morphisms of  $\mathcal{U}_q(\mathfrak{g})$ -modules :

$$\begin{aligned} \overline{S_{1,+} \otimes S_2 \otimes \cdots \otimes S_N \otimes S_{1,-}} & \xrightarrow{\mathcal{I}_N} \overline{S_{1,+} \otimes S_2 \otimes \cdots \otimes S_{N-1} \otimes S_{1,-} \otimes S_N} \\ & \xrightarrow{\mathcal{I}_{N-1}} \cdots \xrightarrow{\mathcal{I}_2} \overline{S_{1,+} \otimes S_{1,-} \otimes S_2 \otimes \cdots \otimes S_N} \simeq S_1 \otimes S_2 \otimes \cdots \otimes S_N. \end{aligned}$$

Note that by construction, each morphism

$$\mathcal{I}_2 \circ \mathcal{I}_3 \circ \cdots \circ \mathcal{I}_j$$

is injective.

As we have established in Lemma 4.2 that  $V$  is cyclic, it suffices to prove that each morphism  $\mathcal{I}_j$  is surjective to get the cyclicity of  $S_1 \otimes \cdots \otimes S_N$ .

Let us start with  $\mathcal{I}_2$ . As  $S_1 \otimes S_2$  is cyclic, the composed map

$$S_{1,+} \otimes S_2 \otimes S_{1,-} \rightarrow S_{1,+} \otimes S_{1,-} \otimes S_2 \rightarrow S_1 \otimes S_2$$

is surjective. In particular  $\mathcal{I}_2$  is an isomorphism of  $\mathcal{U}_q(\mathfrak{g})$ -modules.

Now let us focus on  $\mathcal{I}_3$ . As  $\mathcal{I}_2 \circ \mathcal{I}_3$  is injective and we just proved that  $\mathcal{I}_2$  is an isomorphism,  $\mathcal{I}_3$  is injective and it suffices to establish that  $\mathcal{I}_3$  is surjective. We proceed in two steps : we first establish that a certain deformation of  $\mathcal{I}_3$  is surjective, and in a second step we prove that it implies that  $\mathcal{I}_3$  itself is surjective by a specialization argument.

*First step*

In the same way as for  $\mathcal{I}_2$ , we have an isomorphism

$$\mathcal{J}_3 : \overline{S_2 \otimes S_{1,+} \otimes S_3 \otimes S_{1,-} \otimes S_4 \otimes \cdots \otimes S_N} \rightarrow \overline{S_2 \otimes S_1 \otimes S_3 \otimes \cdots \otimes S_N}$$

---

<sup>5</sup>By abuse of notation, we will use the same symbol for the kernels in several tensor products as it does not lead to confusion.

where the bar means as above that we consider the quotients by the kernels (of the morphisms to  $S_2 \otimes S_1 \otimes S_3 \otimes \cdots \otimes S_N$  this time).

To deduce informations on  $\mathcal{I}_3$  from  $\mathcal{J}_3$ , we consider the morphism

$$\mathcal{I}_{S_{1,+}, S_2} : S_{1,+} \otimes S_2 \rightarrow S_2 \otimes S_{1,+}.$$

This map is not invertible in general, that is why we consider its deformation

$$A(z) = \mathcal{T}_{S_{1,+}, S_2}(z) : S_{1,+} \otimes S_2(z) \rightarrow S_2(z) \otimes S_{1,+}$$

as in section 2.5. It is an isomorphism of  $U_{q,z}(\mathfrak{g})$ -modules, with the action on  $S_2$  twisted by  $\tau_z$ .

Note that the actions on the modules  $S_{1,+}$  and  $S_{1,-}$  are not twisted, so we can use the morphisms  $\mathcal{I}_{\cdot, \cdot}(z)$  as in Remark 2.8. The map  $A(z)$  is compatible with the quotient defining the bar modules above. This means that the kernel in

$$S_{1,+} \otimes S_2(z) \otimes S_3 \otimes S_{1,-} \otimes \cdots \otimes S_N \text{ (resp. in } S_{1,+} \otimes S_2(z) \otimes S_{1,-} \otimes S_3 \otimes \cdots \otimes S_N)$$

is sent by  $A(z) \otimes \text{Id}$  to the kernel in

$$S_2(z) \otimes S_{1,+} \otimes S_3 \otimes S_{1,-} \otimes \cdots \otimes S_N \text{ (resp. in } S_2(z) \otimes S_{1,+} \otimes S_{1,-} \otimes S_3 \otimes \cdots \otimes S_N).$$

Indeed it follows from Remark 2.7 that the following diagram is commutative (the  $\otimes \text{Id}$  are omitted for clarity of the diagram) :

$$\begin{array}{ccccc} & & S_{1,+} S_2(z) S_{1,-} S_3 & \xrightarrow{\mathcal{T}_{S_2, S_{1,-}}(z^{-1})} & S_{1,+} S_{1,-} S_2(z) S_3 & \xrightarrow{\mathcal{I}_{S_{1,+}, S_{1,-}}(z)} & S_1 S_2(z) S_3 . \\ & \nearrow \mathcal{I}_{S_2, S_{1,-}}(z) & & & & & \uparrow \mathcal{T}_{S_2, S_1}(z^{-1}) \\ S_{1,+} S_2(z) S_2 S_{1,-} & & & & & & \\ & \searrow A(z) & & & & & \\ & & S_2(z) S_{1,+} S_3 S_{1,-} & \xrightarrow{\mathcal{T}_{S_3, S_{1,-}}(z)} & S_2(z) S_{1,+} S_{1,-} S_3 & \xrightarrow{\mathcal{I}_{S_{1,+}, S_{1,-}}(z)} & S_2(z) S_1 S_3 \end{array}$$

where we used the relation (3) to inverse  $\mathcal{T}_{S_1, S_2}(z)$ .

So  $A(z)$  induces isomorphisms<sup>6</sup>  $\bar{A}(z)$  :

$$\begin{array}{ccc} \overline{S_{1,+} \otimes S_2(z) \otimes S_3 \otimes S_{1,-} \otimes \cdots \otimes S_N} & \xrightarrow{\mathcal{I}_3(z)} & \overline{S_{1,+} \otimes S_2(z) \otimes S_{1,-} \otimes \cdots \otimes S_N} \\ \downarrow \bar{A}(z) & & \downarrow \bar{A}(z) \\ \overline{S_2(z) \otimes S_{1,+} \otimes S_3 \otimes S_{1,-} \otimes \cdots \otimes S_N} & \xrightarrow{\mathcal{J}_3(z)} & \overline{S_2(z) \otimes S_1 \otimes S_3 \otimes \cdots \otimes S_N} \end{array}$$

satisfying

$$(6) \quad \mathcal{I}_3(z) = (\bar{A}(z))^{-1} \mathcal{J}_3(z) \bar{A}(z).$$

With obvious notations, we have

$$\begin{aligned} \overline{S_2(z) \otimes S_{1,+} \otimes S_3 \otimes S_{1,-} \otimes \cdots \otimes S_N} &\simeq \overline{S_2(z) \otimes \overline{S_{1,+}} \otimes S_3 \otimes S_{1,-} \otimes \cdots \otimes S_N}, \\ \overline{S_2(z) \otimes S_1 \otimes S_3 \otimes \cdots \otimes S_N} &\simeq \overline{S_2(z) \otimes \overline{S_1} \otimes S_3 \otimes \cdots \otimes S_N}. \end{aligned}$$

<sup>6</sup>By abuse of notation with use the same symbol  $\bar{A}(z)$  for them when it does not lead to confusion.

In particular  $\mathcal{J}_3(z)$  is obtained from  $\mathcal{J}_3$  just by the extension of scalars from  $\mathbb{C}$  to  $\mathbb{C}(z)$ . This implies that  $\mathcal{J}_3(z)$  is an isomorphism of  $U_{q,z}(\mathfrak{g})$ -modules.

The conjugation relation (6) implies that  $\mathcal{I}_3(z)$  is also an isomorphism of  $U_{q,z}(\mathfrak{g})$ -modules.

*Second step*

The relation between  $\mathcal{I}_3(z)$  and  $\mathcal{I}_3$  is not as direct as for  $\mathcal{I}_2$  above, that is why we use the ring  $\mathcal{A}$  as in Remark 2.9 in order to define specialization maps. The map  $\mathcal{I}_3(z)$  and the maps used to define the corresponding bar-modules make sense not only over  $\mathbb{C}(z)$  but also over  $\mathcal{A} \subset \mathbb{C}(z)$ , see remark 2.9.  $\mathcal{I}_3(z)$  is obtained just by the extension of scalars from  $\mathcal{A}$  to  $\mathbb{C}(z)$  of an isomorphism of  $\mathcal{A}$ -modules  $\mathcal{I}_{3,\mathcal{A}}(z)$ . So we work with  $\mathcal{A}$ -lattices as defined in remark 2.9 and we consider the specialization map

$\pi : S_{1,+,\mathcal{A}} \otimes_{\mathcal{A}} S_{2,\mathcal{A}}(z) \otimes_{\mathcal{A}} S_{3,\mathcal{A}} \otimes_{\mathcal{A}} S_{1,-,\mathcal{A}} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} S_{N,\mathcal{A}} \rightarrow S_{1,+} \otimes S_2 \otimes S_3 \otimes S_{1,-} \otimes \cdots \otimes S_N$ ,  
taking the quotient by

$$(z-1)S_{1,+,\mathcal{A}} \otimes_{\mathcal{A}} S_{2,\mathcal{A}}(z) \otimes_{\mathcal{A}} S_{3,\mathcal{A}} \otimes_{\mathcal{A}} S_{1,-,\mathcal{A}} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} S_{N,\mathcal{A}}.$$

Then the kernel

$$\text{Ker}_{z,\mathcal{A}} \subset S_{1,+,\mathcal{A}} \otimes_{\mathcal{A}} S_{2,\mathcal{A}}(z) \otimes_{\mathcal{A}} S_{3,\mathcal{A}} \otimes_{\mathcal{A}} S_{1,-,\mathcal{A}} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} S_{N,\mathcal{A}}$$

used to define the bar-module is contained in  $\pi^{-1}(\text{Ker})$  where

$$\text{Ker} \subset S_{1,+} \otimes S_2 \otimes S_3 \otimes S_{1,-} \otimes \cdots \otimes S_N$$

is as in Formula (5). We have the following situation :

$$S_{1,+,\mathcal{A}} \otimes_{\mathcal{A}} S_{2,\mathcal{A}}(z) \otimes_{\mathcal{A}} S_{3,\mathcal{A}} \otimes_{\mathcal{A}} S_{1,-,\mathcal{A}} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} S_{N,\mathcal{A}} \supset \pi^{-1}(\text{Ker}) \supset \text{Ker}_{z,\mathcal{A}}.$$

This is analog for  $S_{1,+,\mathcal{A}} \otimes_{\mathcal{A}} S_{2,\mathcal{A}}(z) \otimes_{\mathcal{A}} S_{1,-,\mathcal{A}} \otimes_{\mathcal{A}} S_{3,\mathcal{A}} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} S_{N,\mathcal{A}}$ . We have a morphism of  $U_{q,\mathcal{A}}(\mathfrak{g})$ -modules

$$\begin{aligned} \mathcal{I}_{3,\mathcal{A}}(z) : S_{1,+,\mathcal{A}} \otimes_{\mathcal{A}} S_{2,\mathcal{A}}(z) \otimes_{\mathcal{A}} S_{3,\mathcal{A}} \otimes_{\mathcal{A}} S_{1,-,\mathcal{A}} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} S_{N,\mathcal{A}} / \text{Ker}_{z,\mathcal{A}} \\ \rightarrow S_{1,+,\mathcal{A}} \otimes_{\mathcal{A}} S_{2,\mathcal{A}}(z) \otimes_{\mathcal{A}} S_{1,-,\mathcal{A}} \otimes_{\mathcal{A}} S_{3,\mathcal{A}} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} S_{N,\mathcal{A}} / \text{Ker}_{z,\mathcal{A}}. \end{aligned}$$

It is surjective after extensions of scalars, that is :

$$(7) \quad \begin{aligned} S_{1,+,\mathcal{A}} \otimes_{\mathcal{A}} S_{2,\mathcal{A}}(z) \otimes_{\mathcal{A}} S_{1,-,\mathcal{A}} \otimes_{\mathcal{A}} S_{3,\mathcal{A}} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} S_{N,\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{C}(z) \\ = \text{Ker}_{z,\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{C}(z) + \text{Im}(\tilde{\mathcal{I}}_{3,\mathcal{A}}(z)) \otimes_{\mathcal{A}} \mathbb{C}(z), \end{aligned}$$

where  $\tilde{\mathcal{I}}_{3,\mathcal{A}}(z)$  is defined as  $\mathcal{I}_{3,\mathcal{A}}(z)$  :

$$\begin{aligned} \tilde{\mathcal{I}}_{3,\mathcal{A}}(z) : S_{1,+,\mathcal{A}} \otimes_{\mathcal{A}} S_{2,\mathcal{A}}(z) \otimes_{\mathcal{A}} S_{3,\mathcal{A}} \otimes_{\mathcal{A}} S_{1,-,\mathcal{A}} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} S_{N,\mathcal{A}} \\ \rightarrow S_{1,+,\mathcal{A}} \otimes_{\mathcal{A}} S_{2,\mathcal{A}}(z) \otimes_{\mathcal{A}} S_{1,-,\mathcal{A}} \otimes_{\mathcal{A}} S_{3,\mathcal{A}} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} S_{N,\mathcal{A}}. \end{aligned}$$

Note that

$$\begin{aligned} (z-1)^{\mathbb{Z}} \text{Ker}_{z,\mathcal{A}} \cap (S_{1,+,\mathcal{A}} \otimes_{\mathcal{A}} S_{2,\mathcal{A}}(z) \otimes_{\mathcal{A}} S_{1,-,\mathcal{A}} \otimes_{\mathcal{A}} S_{3,\mathcal{A}} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} S_{N,\mathcal{A}}) = \text{Ker}_{z,\mathcal{A}}, \\ (z-1)^{\mathbb{Z}} \text{Im}(\tilde{\mathcal{I}}_{3,\mathcal{A}}(z)) \cap (S_{1,+,\mathcal{A}} \otimes_{\mathcal{A}} S_{2,\mathcal{A}}(z) \otimes_{\mathcal{A}} S_{1,-,\mathcal{A}} \otimes_{\mathcal{A}} S_{3,\mathcal{A}} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} S_{N,\mathcal{A}}) = \text{Im}(\tilde{\mathcal{I}}_{3,\mathcal{A}}(z)). \end{aligned}$$

This is clear for the kernel, and the image of  $\tilde{\mathcal{I}}_{3,\mathcal{A}}(z)$  is obtained just by the scalar extension from  $\mathbb{C}$  to  $\mathcal{A}$ .

Consequently, Equation (7) implies

$$(8) \quad S_{1,+,\mathcal{A}} \otimes_{\mathcal{A}} S_{2,\mathcal{A}}(z) \otimes_{\mathcal{A}} S_{1,-,\mathcal{A}} \otimes_{\mathcal{A}} S_{3,\mathcal{A}} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} S_{N,\mathcal{A}} = \text{Ker}_{z,\mathcal{A}} + \text{Im}(\tilde{\mathcal{I}}_{3,\mathcal{A}}(z)).$$

Indeed by (7),  $\lambda$  in the left hand side can be written as

$$\lambda = (z-1)^{-a}\alpha + (z-1)^{-b}\beta$$

where  $a, b$  are minimal non-negative integers such that  $\alpha \in \text{Ker}_{z, \mathcal{A}}$  and  $\beta \in \text{Im}(\tilde{\mathcal{I}}_{3, \mathcal{A}}(z))$ . If  $a, b > 0$ , this contradicts the hypothesis on  $\lambda$ . If  $a = 0$ , then  $(z-1)^{-b}\beta$  is in the lattice and so  $b = 0$ . The argument is identical for  $b = 0$ . This establishes the identity.

Now by applying the map  $\pi$  to Equation (8), we get

$$S_{1,+} \otimes S_2 \otimes S_{1,-} \otimes S_3 \otimes \cdots \otimes S_N = \pi(\text{Ker}_{z, \mathcal{A}}) + \pi(\text{Im}(\tilde{\mathcal{I}}_{3, \mathcal{A}}(z))),$$

and so the surjectivity of

$$\begin{aligned} \mathcal{I}_3 : S_{1,+,\mathcal{A}} \otimes_{\mathcal{A}} S_{2,\mathcal{A}}(z) \otimes_{\mathcal{A}} S_{3,\mathcal{A}} \otimes_{\mathcal{A}} S_{1,-,\mathcal{A}} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} S_{N,\mathcal{A}} / \pi^{-1}(\text{Ker}) \\ \rightarrow S_{1,+,\mathcal{A}} \otimes_{\mathcal{A}} S_{2,\mathcal{A}}(z) \otimes_{\mathcal{A}} S_{1,-,\mathcal{A}} \otimes_{\mathcal{A}} S_{3,\mathcal{A}} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} S_{N,\mathcal{A}} / \pi^{-1}(\text{Ker}). \end{aligned}$$

We have established that  $\mathcal{I}_3$  is an isomorphism.

In the same way, we prove that  $\mathcal{I}_j$  is an isomorphism by induction on  $j \geq 3$ .

**Remark 4.3.** *As discussed in the introduction, the result in this paper implies the main result of [H3]. However, the author would like to clarify<sup>7</sup> the end of the direct proof in [H3, Section 6]. Let us use the notations of that paper. As for the bar-modules above, the tensor products there have to be understood up to the kernel of the morphisms to  $S_1 \otimes \cdots \otimes S_N$ . By the same deformation arguments as above, the surjectivity of the morphism*

$$S_N^+ \otimes S_i \otimes S_N^- \rightarrow S_i \otimes S_N$$

*implies the surjectivity at the level of bar-modules of the maps considered in the final arguments of [H3, Section 6].*

**4.3. Examples.** Let us give some examples to illustrate some crucial steps in the proof.

For all examples below we set  $\mathfrak{g} = \hat{sl}_2$ .

*Example 1* : let us set

$$S_1 = L(Y_1 Y_{q^2}), \quad S_2 = V(1), \quad S_3 = V(1).$$

For  $i > j$ , the  $S_i \otimes S_j$  are simple. We have

$$S_{1,+} = V(q^2)$$

and the kernel of  $\mathcal{I}_{S_1,+, S_2}$  is isomorphic to the trivial module of dimension 1. In particular :

$$\begin{aligned} V(q^2) \otimes V(1)^{\otimes 3} \supset \text{Ker} \simeq V(1)^{\otimes 2}. \\ V(q^2)_{\mathcal{A}} \otimes_{\mathcal{A}} V_{\mathcal{A}}(z) \otimes_{\mathcal{A}} V(1)_{\mathcal{A}}^{\otimes 2} \supset \text{Ker}_{z, \mathcal{A}} \simeq V_{\mathcal{A}}(z) \otimes_{\mathcal{A}} V(1)_{\mathcal{A}}. \end{aligned}$$

We have the deformed operator

$$A(z) : V(q^2)_{\mathcal{A}} \otimes_{\mathcal{A}} V_{\mathcal{A}}(z) \rightarrow V_{\mathcal{A}}(z) \otimes_{\mathcal{A}} V(q^2)_{\mathcal{A}}$$

which is of the form

$$A(z) = A + \mathcal{O}(z-1)$$

---

<sup>7</sup>M. Gurevich, E. Lapid and A. Minguez asked a question about an argument in [H3] which had to be clarified. This is done in this Remark 4.3.

where  $rk(A) = 3$  with

$$A^{-1}(z) = \frac{A'}{z-1} + \mathcal{O}(1)$$

where  $rk(A') = 1$  (we use the standard asymptotical comparison notation  $\mathcal{O}$ ). We have the isomorphism  $\mathcal{J}_3, \mathcal{I}_3$  induced from  $\mathcal{I}_{V(1),V(1)}$  :

$$\begin{array}{ccc} \overline{V(q^2) \otimes V(z) \otimes V(1) \otimes V(1)} & \xrightarrow{\mathcal{I}_3(z)} & \overline{V(q^2) \otimes V(z) \otimes V(1) \otimes V(1)} \\ \downarrow \overline{A(z)} & & \downarrow \overline{A(z)} \\ \overline{V(z) \otimes V(q^2) \otimes V(1) \otimes V(1)} & \xrightarrow{\mathcal{J}_3(z)} & \overline{V(z) \otimes V(q^2) \otimes V(1) \otimes V(1)} \end{array}$$

If had set instead  $S_2 = V(q^2)$ ,  $S_2 \otimes S_3$  would be cyclic but not simple. Nothing would change, except that  $A(0)$  would be an isomorphism and the inverse  $(A(z))^{-1}$  would be regular at 0.

*Example 2* : let us set

$$S_1 = L(Y_1 Y_{q^2} Y_{q^4}^2) \simeq L(Y_1 Y_{q^2} Y_{q^4}) \otimes V(q^4), \quad S_2 = V(q^2), \quad S_3 = V(1).$$

$S_1 \otimes S_3$  is simple.  $S_1 \otimes S_3$  and  $S_2 \otimes S_3$  are cyclic (but not simple). We have  $S_{1,+} = L(Y_{q^2} Y_{q^4}^2) \simeq L(Y_{q^2} Y_{q^4}) \otimes V(q^4)$  and

$$L(Y_{q^2} Y_{q^4}^2) \otimes V(q^2) \otimes V(1)^{\otimes 2} \supset \text{Ker} \simeq M \otimes V(1)$$

where  $M$  is of length 2 with simple constituents isomorphic to  $L(Y_{q^2} Y_{q^4}^2)$  and  $V(q^4)$  (this follows from the study of the module  $L(Y_{q^2} Y_{q^4}^2) \otimes V(q^2) \otimes V(1)$  of length 4).

$$(L(Y_{q^2} Y_{q^4}^2))_{\mathcal{A}} \otimes_{\mathcal{A}} V(q^2 z)_{\mathcal{A}} \otimes V(1)_{\mathcal{A}}^{\otimes 2} \supset \text{Ker}_{z,\mathcal{A}} = V(q^4)_{\mathcal{A}}^{\otimes 2} \otimes_{\mathcal{A}} V(q^2 z)_{\mathcal{A}} \otimes_{\mathcal{A}} V(1)_{\mathcal{A}}.$$

We have

$$A(z) : L(Y_{q^2} Y_{q^4}^2) \otimes V(q^2 z) \rightarrow V(q^2 z) \otimes L(Y_{q^2} Y_{q^4}^2)$$

which is obtained from  $\mathcal{T}_{V(q^4),V(q^2)}(z)$  and  $\mathcal{T}_{L(Y_{q^2} Y_{q^4}),V(q^4)}(z)$  with  $L(Y_{q^2} Y_{q^4}) \otimes V(q^4)$  simple. Hence we have  $A(z) = A + \mathcal{O}(z-1)$  where  $rk(A) = 12$  and  $A^{-1}(z) = \frac{A'}{z-1} + \mathcal{O}(1)$  where  $rk(A') = 4$ . As above the isomorphisms  $\mathcal{J}_3, \mathcal{I}_3$  are induced from  $\mathcal{I}_{V(1),V(1)}$ .

*Example 3* : let us set

$$S_1 = L(Y_1 Y_{q^2} Y_{q^4}), \quad S_2 = V(q^4), \quad S_3 = V(q^2).$$

The tensor products  $S_1 \otimes S_2$  and  $S_1 \otimes S_3$  are simple. The tensor product  $S_2 \otimes S_3$  is cyclic (but not simple). We have  $S_{1,+} = L(Y_{q^2} Y_{q^4})$  and

$$\text{Ker}_{z,\mathcal{A}} \simeq V(q^4) \otimes_{\mathcal{A}} V(q^4 z) \otimes_{\mathcal{A}} V(q^2).$$

We use

$$A(z) : L(Y_{q^2} Y_{q^4}) \otimes V(q^4 z) \rightarrow V(q^4 z) \otimes L(Y_{q^2} Y_{q^4})$$

with  $A(0)$  which is an isomorphism. The inverse  $(A(z))^{-1}$  is regular at 0. We have the morphisms  $\mathcal{J}_3, \mathcal{I}_3$  :

$$\begin{array}{ccc} \overline{L(Y_{q^2}Y_{q^4}) \otimes V(zq^4) \otimes V(q^2) \otimes V(1)} & \xrightarrow{\mathcal{I}_3(z)} & \overline{L(Y_{q^2}Y_{q^4}) \otimes V(q^4z) \otimes V(1) \otimes V(q^2)} . \\ \downarrow \overline{A(z)} & & \downarrow \overline{A(z)} \\ \overline{V(zq^4) \otimes L(Y_{q^2}Y_{q^4}) \otimes V(q^2) \otimes V(1)} & \xrightarrow{\mathcal{J}_3(z)} & \overline{V(q^4z) \otimes L(Y_{q^2}Y_{q^4}) \otimes V(1) \otimes V(q^2)} \end{array}$$

Note that

$$\begin{aligned} \overline{V(zq^4) \otimes L(Y_{q^2}Y_{q^4}) \otimes V(q^2) \otimes V(1)} &\simeq \overline{V(zq^4) \otimes L(Y_1Y_{q^2}Y_{q^4}) \otimes V(q^2)} \\ &\simeq \overline{V(q^4z) \otimes L(Y_{q^2}Y_{q^4}) \otimes V(1) \otimes V(q^2)} \end{aligned}$$

and  $\mathcal{J}_3(z)$  is an isomorphism. This module is simple for generic  $z$ , and so  $\mathcal{I}_3(z)$  is an isomorphism as well.

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